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Imminant polynomials of graphs

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Summary. The imminant polynomials of the adjacency matrices of graphs are defined. The imminant polynomials of several graphs [linear graphs (L_n), cyclic graphs (C_n) and complete graphs (K_n)] are obtained. It is shown that the characteristic polynomials and permanent polynomials become special cases of imminant polynomials. The connection between the Schur-functions and imminant polynomials is outlined.

Key words: Graphs – Imminant polynomials – Schur-functions

1 Introduction

Quest for graph-theoretical (structural) invariants has a long history in mathematical and chemical literature [1–25]. For example, computer representation of a molecular structure using the underlying connection table is not unique since it is label independent. It was observed by several workers (see for example, Ref. [24]) that the determinant of the connection table (adjacency matrix of the associated graph) and the characteristic polynomial of the graph are invariant to labeling. Consequently, it was thought for some time that the characteristic polynomials of graphs are unique structural invariants. However, this early conjecture is now well-known to be false since one could produce two non-isomorphic graphs with the same characteristic polynomials [24].

The characteristic polynomials of graphs [$\det(xI - A)$ where A is the adjacency matrix] have numerous other chemical applications [1–21] although perhaps they were explored in depth for the first time in the chemical context of Hückel theory. While characteristic polynomials have received considerable attention in the chemical literature [1–21], this is not the case with other structural invariants. For example, the related permanent polynomial $\mathrm{Per}(xI - A)$ has received some attention in the mathematical literature [22–24]. In fact, it has been shown that the permanental polynomials discriminate some isospectral graphs when characteristic polynomials fail to differentiate those [23].

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Nevertheless, there are trees for which the permanental polynomials are the same and thus in general, the permanental polynomials are not unique structural invariants.

The imminant polynomials proposed here combine the character theory of symmetric group S_n with graph theory. It is likely that these polynomials thus would have powerful and more significant applications than the ordinary characteristic polynomials which certainly are known to have numerous applications in several areas ranging from quantum chemistry to chemical kinetics. If the imminant polynomials can be generated through alternative routes then they would serve as the generators of the characters of the S_n group which are very important in chemistry and physics.

In this manuscript, we put forth more general structural invariants which we call the imminant polynomials of graphs. Both the characteristic and permanent polynomials discussed in the literature up to now become special cases of the imminant polynomials. The imminant polynomials use the characters of the symmetric groups S_n (the group of $n!$ permutations of n objects). The relationship between the imminant polynomials and the Schur-functions (S-functions) is also discussed. The imminant polynomials of linear graphs (L_n), cyclic graph (C_n) and complete graphs (K_n) are tabulated.

2 Imminant polynomials of graphs

Let A be the adjacency matrix of a graph containing n vertices defined as follows:

$$A_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and the vertices } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise, } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant n \end{cases}$$

Consider the matrix A' defined by $(-xI + A)$. Note that equivalently, one may consider the matrix $A - xI$. Suppose a_{ij}'s are the matrix elements of the $(-xI + A)$ matrix. Let s be a permutation in the group S_n (recall n is the number of vertices) described in the permutation notation as:

$$s = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ e_1 & e_2 & e_3 & \cdots & e_n \end{bmatrix}$$

(That is, 1 goes to e_1, 2 goes to $e_2 \ldots n$ goes to e_n.) Consider the product P_s defined by:

$$P_s = a_{1e_1} a_{2e_2} \cdots a_{ne_n}$$

It may be seen that if s is the identity permutation e then:

$$P_e = a_{11} a_{22} \cdots a_{nn}$$

This would simply be the product of the diagonal elements of the matrix $(-xI + A)$. The irreducible representations of the group S_n are well known to be characterized by the partitions of the integer n [25–30], which we will denote by $[\lambda]$. For example, the irreducible representations of the group S_5 are given by $[5]$, $[41]$, $[32]$, $[31^2]$, $[21^2]$, $[21^3]$, $[1^5]$. Thus in general λ is a partition of n defined by $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 1$.

The imminant polynomial of the graph G, associated with the irreducible representation $[\lambda]$ of the group S_n is given by:

$$P_G^{[\lambda]} = \sum_s \chi^{\lambda}(s) P_s$$

where the sum is over all possible $n!$ permutations (s) of n objects. Of course, there are $n!$ such terms and thus the computation of imminants in general is a $n!$ problem. However, as we will show special tricks reduce the number of computations by several orders thereby making computations of imminants a feasible problem for some special cases.

Let $[n]$ be the identity irreducible representation of the symmetric group S_n. In this case $\chi^{\lambda}(s)$ is 1 for all s and thus:

$$P_G^{[n]} = \sum_s P_s.$$

This is precisely the permanent polynomial of a graph studied by Turner [23] in 1968 and revisited by Merris et al. [22] in 1981. Yet there are very few studies on the permanent polynomials of graphs compared to the characteristic polynomials.

Suppose $\lambda = (1, 1, \ldots 1, 1)$ then the imminant polynomials become:

$$P_G^{[1^n]} = \sum_s (-1)^s P_s$$

where $(-1)^s$ is -1 if the permutation s is odd and $+1$ if it is even. It is easy to determine if s is even or odd from its cycle representation. Suppose s contains v_1 cycles of length 1, v_2 cycles of length 2, $\ldots$ v_n cycles of length n then it can be seen that:

$$\begin{aligned} P_G^{[1^n]} &= \sum_s (-1)^{v_2+v_4+v_6+\cdots+v_c} P_s \\ &= \sum_s (-1)^{v_2+v_4+v_6+\cdots+v_c} s(a_{11}a_{22}\cdots a_{nn}) \end{aligned}$$

where $c = n$ if n is even and $(n-1)$ if n is odd. The above expansion is nothing but the determinant of the matrix and thus $P_G^{[1^n]}$ is the characteristic polynomial of the graph.

Except for the special cases namely, $[n]$ and $[1^n]$, the imminant polynomials of G are new and there appears to be no simple computational procedure to generate $P_G^{[\lambda]}$.

For both $[n]$ and $[1^n]$ the imminant polynomials can be generated using the Sachs theorem. The Sachs theorem requires finding all cyclic subgraphs (circuits) and isolated edges in a given graph. Suppose $k(H)$ is the number of such subgraphs. Then Merris et al. [22] show that the absolute values of the coefficients in the permanent polynomial c_i are generated by:

$$c_i = \sum_H 2^{k(H)} \qquad 1 \leqslant i \leqslant n$$

where the sum is over all subgraphs H on i vertices whose components are circuits and isolated edges. In this case, the permanent polynomial is given by:

$$P_G^{[n]} = x^n - c_1 x^{n-1} + c_2 x^{n-2} + \cdots + (-1)^n c_n$$

A similar procedure exists for $P_G^{[1^n]}$ (see for example Ref. [16]).

The current author [3, 4] formulated an efficient technique suitable for computer implementation for the characteristic polynomials of graphs. The method which he called the Frame method was shown by Trinajstić and coworkers [12] due to La Verrier. The current author [3, 4] has shown that using this method characteristic polynomials of graphs containing a large number of vertices can be easily and routinely obtained.

In general, at present there appears to be no general and efficient computational algorithm for the computation of other imminant polynomials. We present here some techniques which simplify the $n!$ nature of the problem. First we consider the generation of $\chi^{(\lambda)}$ itself using the Schur-function method. In order to make this discussion self-contained we summarized below the salient points of the theory of S-functions from Refs. [25–29].

Suppose s_r is a symmetric function of the quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$, defined by:

$$s_r = \sum_{i=1}^{n} \alpha_i^r.$$

Let Z_r be the matrix defined by:

$$[Z_r] = \begin{bmatrix} s_1 & 1 & 0 & 0 & & \ldots & 0 \\ s_2 & s_1 & 2 & 0 & & \ldots & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \ldots & 0 \\ \vdots & & & & & & \\ s_{r-1} & s_{r-2} & & \ldots & & s_1 & r-1 \\ s_2 & s_{r-1} & & \ldots & & s_2 & s_1 \end{bmatrix}$$

Suppose (λ) is a partition $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_p)$ of r with p components (in descending order). The Schur function also known as the S-function $\{\lambda\}$ is defined by the following expression:

$$\{\lambda\} = \frac{1}{r!} |Z_r|^{(\lambda)}$$

where $|Z_r|^{(\lambda)}$ is the imminant of the matrix Z_r associated with the partition (λ). Let h_r and a_r denote the S-functions which correspond to the partitions (r) and (1^r), respectively. The above expression for the S-function $\{\lambda\}$ can be reduced to another convenient form. Let $|C|$ be the order of conjugacy class C of the group S_n and let $\chi_C^{(\lambda)}$ be the character of $[\lambda]$ which corresponds to the class C. Then it can be shown that:

$$\{\lambda\} = \frac{1}{r!} \sum_C |C| \chi_C^{(\lambda)} s_C$$

where s_C is defined by:

$$s_C = s_1^{b_1} s_2^{b_2} s_3^{b_3} \cdots$$

if an element in the conjugacy class C has b_1 cycles of length 1, b_2 cycles of length 2, etc. $|C|$ can be obtained by Cayley's counting principle as:

$$|C| = \frac{r!}{1^{b_1} b_1! 2^{b_2} b_2! \cdots}.$$

S-functions can be illustrated with examples from the group S_4. The group S_4 has five irreducible representations associated with partitions (4), (3, 1), (2, 2), $(2, 1^2)$ and (1^4). From the character table of S_4 we find that:

$$\{4\} = \tfrac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 6s_4 + 3s_2^2)$$

$$\{31\} = \tfrac{1}{24}(3s_1^4 + 6s_1^2 s_2 - 6s_4 - 3s_2^2)$$

$$\{2^2\} = \tfrac{1}{24}(2s_1^2 - 8s_1s_3 + 6s_2^2)$$

$$\{21^2\} = \tfrac{1}{24}(3s_1^4 - 6s_1^2s_2 + 6s_4 - 3s_2^2)$$

$$\{1^4\} = \tfrac{1}{24}(s_1^4 - 6s_1^2s_2 + 8s_1s_3 - 6s_4 + 3s_2^2)$$

S-functions can be obtained as quotient of determinants using the Frobenius formula. Let:

$$\Delta(\alpha_1, \alpha_2, \ldots, \alpha_m) = \prod (\alpha_r - \alpha_s)(r < s) = \sum \pm \alpha_1^{m-1}\alpha_2^{m-2}\cdots\alpha_{m-1}$$

The Frobenius formula which relates s_C, $\Delta(\alpha_1, \ldots, \alpha_n)$ and the character is shown below.

$$s_C\,\Delta(\alpha_1, \alpha_2, \ldots, \alpha_m) = \sum \pm \chi_C^{(\lambda)}\alpha_1^{\lambda_1+n-1}\alpha_2^{\lambda_2+n-2}\cdots\alpha_n^{\lambda_n}$$

From this the S-function can be obtained as:

$$\{\lambda\} = \frac{1}{r!}\sum |C|\chi_C^{(\lambda)} s_C = \frac{\sum \pm \prod_i \alpha_i^{\lambda_1+n-1}}{\sum \pm \prod_i \alpha_i^{n-1}}$$

if the conjugacy class C contains b_1 cycles of length 1, b_2 cycles of length 2, ..., etc. The summation is taken with respect to all permutations, the negative sign is for odd permutations.

Generating functions can also be obtained for S-functions. Let $F(x) = 1 + \sum h_r x^r$ where h_r is the S-function which corresponds to the partition (r). Consider the S-function of the form $\{n; p_1, p_2, \ldots, p_i\}$ with $n \geqslant p_1 \geqslant p_2 > \cdots \geqslant p_i$. Let $g(x)$ be defined as:

$$g(x) = \begin{vmatrix} x^i & x^{i-1} & \cdots & 1 \\ h_{p_1-1} & h_{p_1} & \cdots & h_{p_1+i-1} \\ h_{p_2-2} & h_{p_2-1} & \cdots & h_{p_2+i-2} \\ \vdots & \vdots & & \vdots \\ h_{p_i-i} & h_{p_i-i+1} & \cdots & h_{p_i} \end{vmatrix}$$

$F(x)g(x)$ is a generating function for S-functions of the form $\{n; p_1, p_2, \ldots, p_i\}$. The coefficient of x^{n+i} in $F(x)g(x)$ gives $\{n; p_1, p_2, \ldots, p_i\}$. This method of computing the S-functions amounts to finding the cycle indices of smaller groups. Let P_{S_i} be the cycle index of the symmetric group S_i. The S-function $\{n; p_1, p_2, \ldots, p_n\}$ is the determinant, $\det(P_{S_{p_i-i+j}})$, with the convention $P_{S_0} = 1$ and $P_{S_{-\ell}} = 0$ for a positive integer ℓ [31]. Let us illustrate this with the S-function $\{6; 4, 1, 1\}$. This is shown below as a determinant:

$$\{6; 4, 1, 1\} = \begin{vmatrix} P_{S_4} & P_{S_5} & P_{S_6} \\ P_{S_0} & P_{S_1} & P_{S_2} \\ 0 & P_{S_0} & P_{S_1} \end{vmatrix}$$

$$= P_{S_1}^2 P_{S_4} - P_{S_2}P_{S_4} - P_{S_1}P_{S_5} + P_{S_6}$$

with,

$$P_{S_1} = s_1$$
$$P_{S_2} = \tfrac{1}{2}(s_1^2 + s_2)$$
$$P_{S_4} = \tfrac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 + 6s_4)$$
$$P_{S_5} = \tfrac{1}{120}(s_1^5 + 10s_1^3 s_2 + 20s_1^2 s_3 + 15s_1 s_2^2 + 30s_1 s_4 + 20s_2 s_3 + 24s_5)$$
$$P_{S_6} = \tfrac{1}{720}(s_1^6 + 15s_1^4 s_2 + 40s_1^3 s_3 + 45s_1^2 s_2^2 + 90s_1^2 s_4 + 120s_1 s_2 s_3 + 144s_1 s_5 + 15s_2^3 + 90s_2 s_4 + 40s_3^2 + 120s_6)$$

Substituting these expressions in the determinant expansion we find that:

$$\{6; 4, 1, 1\} = \tfrac{1}{720}(10s_1^6 + 30s_1^4 s_2 + 40s_1^3 s_3 - 90s_1^2 s_2^2 - 120s_1 s_2 s_3 - 30s_2^3 + 40s_3^2 + 120s_6)$$

Using the techniques S-functions characters of groups of large orders can be obtained. Liu and the author [30] have devised a computer code which generates the character tables of the symmetric groups S_n for n up to 20. For larger n it is still possible to generate selected characters since generation of all characters becomes overwhelming. In our method, computation of the imminant polynomials requires as the first step computation of the S-functions associated with $[\lambda]$. It seems to the author that there may be a direct way of using the Frobenius determinant method to compute imminant polynomials using the permanent polynomials of smaller graphs. However, this is speculative at present. Further work is warranted in this direction.

The second step is to identify surviving products for each permutation s in a given conjugacy class. It is clear that the power of x in the imminant polynomial is determined by the number 1-cycles in the permutation s. If there are v_1 1-cycles in the permutation s then the power of x is simply v_1. We group together conjugacy classes which have the same number of 1-cycles. Then identify the surviving products under that permutation. For each such group of permutations we compute the surviving terms multiplied by the character and sum over all the conjugacy classes. It is important to emphasize that all permutations in a conjugacy class will not lead to non-zero contributions, in general, for any graph. Only for a complete graph K_n do permutations in a given conjugacy class yield the same contribution.

We now illustrate the procedure with a few examples. Consider first the line graph L_4 in Fig. 1. The conjugacy class 1^4 yields the x^4 term for all imminant polynomials. The x^2 term for all graphs containing 4 vertices is generated by 6 permutations in the conjugacy class $1^2 2$. However it is readily seen that only three of these permutations (namely, (12)(3)(4), (23)(1)(4), (1)(2)(34)) lead to non-zero terms for L_4. Hence the coefficient of x^2 in the imminant is three times the character corresponding to $1^2 2$. It is easily seen that 8 permutations of the type 13 lead to zero terms and thus the coefficients of x terms are zero in all imminants. The constant terms are generated by two conjugacy classes of the

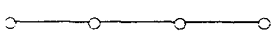

Fig. 1. A linear graph on 4 vertices (L_4)

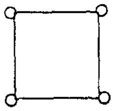

Fig. 2. A cyclic graph on 4 vertices (C_4)

type 4 and 2^2. Of these only one term in 2^2 survives [(12)(34)] for the L_4 graph. Thus the five imminant polynomials of the L_4 graph are:

$$P_G^{[4]} = x^4 + 3x^2 + 1$$

$$P_G^{[31]} = 3x^4 + 3x^2 - 1$$

$$P_G^{[2^2]} = 2x^4 + 2$$

$$P_G^{[21^2]} = 3x^4 - 3x^2 - 1$$

$$P_G^{[1^4]} = x^4 - 3x^2 + 1$$

As another example consider the C_4 graph in Fig. 2. Consider the [31] irreducible representation. For this graph four of the six $1^2 2$ type permutations survive, none of 13 type permutations survive, 2 of six permutations of the type 4 survive (namely (1234), (1432)), and 2 of 3 permutations of the 2^2 type survive [(12)(34), (14)(23)]. Hence the [3, 1] imminant polynomial is:

$$P_G^{[3,1]} = 3x^4 + 4x^2 - 4$$

Graphs of various complexities can be treated this way. Consider the graph in Fig. 3. Note that this graph is related to C_4 by addition of another diagonal. All imminant polynomials of this graph are as follows:

$$P_G^{[4]} = x^4 + 5x^2 - 4x + 4$$

$$P_G^{[3,1]} = 3x^4 + 5x^2 - 4$$

$$P_G^{[2^2]} = 2x^4 + 4x + 4$$

$$P_G^{[21^2]} = 3x^4 - 5x^2$$

$$P_G^{[1^6]} = x^4 - 5x^2 - 4x$$

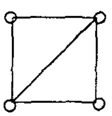

Fig. 3. A graph on 4 vertices

3 Imminant polynomials of certain classes of graphs

In this section we compute the imminant polynomials of a series of graphs. We first consider the linear graphs L_n. Table 1 lists all the computed imminant polynomials of L_n graphs for $n = 4 - 6$. For all linear graphs it can be seen from Table 1 that the magnitudes of the coefficients of the permanent polynomial and the characteristic polynomial are the same. As a matter of fact, Merris et al. [22]

Table 1. Imminant polynomials of line graphs L_n ($n = 4 - 6$)

Γ	L_4 x^4	x^2	1
[4]	1	3	1
[31]	3	3	−1
[2²]	2	0	2
[21²]	3	−3	−1
[1⁴]	1	−3	1

Γ	L_5 x^5	x^3	x
[5]	1	4	3
[41]	4	8	0
[32]	5	4	3
[31²]	6	0	−6
[2²1]	5	−4	3
[21³]	4	−8	0
[1⁵]	1	−4	3

Γ	L_6 x^6	x^4	x^2	1
[6]	1	5	6	1
[51]	5	15	6	−1
[42]	9	15	6	3
[41²]	10	10	−12	−2
[3²]	5	5	6	−3
[321]	16	0	0	0
[2³]	5	−5	6	3
[31³]	10	−10	−12	2
[2²1²]	9	−15	6	−3
[21⁴]	5	−15	6	1
[1⁶]	1	−5	6	−1

have proven that: suppose T is any tree graph and if $A(T)$ is the adjacency matrix and let:

$$\det[xI - A(T)] = x^n - a_1x^{n-1} + a_2x^{n-2} + \cdots + (-1)^n a_n$$

$$\operatorname{per}[xI - A(T)] = x^n - c_1x^{n-1} + c_2x^{n-2} + \cdots + (-1)^n c_n$$

then

$$c_i = |a_i| \qquad \forall 1 \leqslant i \leqslant n$$

This result can be proven using the Sachs theorem since for all tree graphs $c_i = a_i = 0$ for odd i and for even i the number of single edges is $i/2$.

The characteristic polynomial (imminant polynomial which corresponds to $[1^n]$) of the line graph L_n is given by the Chebyshev polynomial [11]:

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k}$$

Hence we conclude that the coefficients all permanental polynomials of L_n graphs are generated by the Chebychev polynomials.

As seen from Table 1 for the L_n graphs, the magnitudes of the coefficients of the imminant polynomial $[\lambda]$ are the same as its conjugate $[\lambda]^\dagger$. For example, the irreducible representations [31] and $[21^2]$ are conjugates for S_4. Likewise the representations [41], and $[21^3]$ are conjugates, [32] and $[2^2 1]$, etc., are conjugates. The magnitudes of the coefficients of their imminant polynomials are the same (Table 1).

Table 2 shows all imminant polynomials of the cyclic graphs C_n (for example, see Fig. 2 for C_4). In general, there is no simple relation between the imminant polynomial $P_G^{[\lambda]}$ and the imminant polynomial of its conjugate $P_G^{[\lambda]\dagger}$. For example, the characteristic polynomial of the C_4 graph is given by:

$$P_G^{[1^4]} = x^4 - 4x^2$$

Table 2. Imminant polynomials of cyclic graphs C_n $(n = 4 - 6)$

C_4				
Γ	x^4	x^2	1	
[4]	1	4	4	
[31]	3	4	−4	
$[2^2]$	2	0	4	
$[21^2]$	3	−4	0	
$[1^4]$	1	−4	0	
C_5				
Γ	x^5	x^3	x	1
[5]	1	5	5	−2
[41]	4	10	0	2
[32]	5	5	0	0
$[31^2]$	6	0	0	−2
$[2^2 1]$	5	−5	0	0
$[21^3]$	4	−10	0	2
$[1^5]$	1	−5	5	−2
C_6				
Γ	x^6	x^4	x^2	1
[6]	1	6	9	4
[51]	5	18	9	−4
[42]	9	18	9	6
$[41^2]$	10	12	−18	−2
$[3^2]$	5	6	9	−6
[321]	16	0	0	0
$[2^3]$	5	−6	9	6
$[31^3]$	10	−12	−18	2
$[2^2 1^2]$	9	−18	9	−6
$[21^4]$	5	−18	9	4
$[1^6]$	1	−6	9	−4

Likewise the imminant polynomials of [31] and [21²] conjugate representations are:

$$P_G^{[31]} = 3x^4 + 4x^2 - 4$$
$$P_G^{[21^2]} = 3x^4 - 4x^2$$

For the C_5 graph, however, the coefficients of the imminant polynomials of conjugate partitions differ only in signs.

The imminant polynomial corresponding to the $[1^n]$ representation of the C_n graph can be obtained in terms of the linear graphs using the well-known recursive relation [11]:

$$P_{C_n}^{[1^n]} = P_{L_n}^{[1^n]} - P_{L_{n-2}}^{[1^{n-2}]} + (-1)^n 2$$

We found similar relations to hold for other imminant polynomials as well. For example the following relationship is true:

$$P_{C_6}^{[5,1]} = P_{L_6}^{[5,1]} + P_{L_4}^{[3,1]} - 2$$

For other imminant polynomials, relationships between C_n and L_n graphs appear to be more complicated.

Next we consider the complete graphs on n vertices denoted by K_n. An example of this graph namely, the K_4 graph is shown in Fig. 4. Table 3 shows the imminant polynomials of the K_n graphs for $n = 4$–7. The computation of the imminant polynomials of K_n is challenging in the sense that none of the $n!$ terms cancel out since the graph is complete. On the other hand, the S-functions can be used to full advantage since every member in a conjugacy class makes the same contribution although different conjugacy classes of S_n can also yield the same power of x. First we note that there is no simple relationship between the coefficients of $P_G^{[\lambda]}$ and $P_G^{[\lambda]^\dagger}$ for the K_n graph. In particular, the permanental polynomial $P_G^{[n]}$ and $P_G^{[1^n]}$ are not simply related for the complete graph K_n.

It is known in the literature that the characteristic polynomial of the complete graph K_n [11, 24] is given by:

$$P_{K_n}^{[1^n]} = (x - n + 1)(1 + x)^{n-1}$$

It can be seen from Table 3 that all our results for the $[1^n]$ representation conform to this although we did not use this fact in computing the results in Table 3.

The imminant polynomials of the K_n graphs of each imminant differ and are sufficiently complex. This gives us hope that it is conceivable that other imminant polynomials may differ for two non-isomorphic graphs whose characteristic polynomials are the same (isospectral or cospectral graphs). For example, Harary et al. [24] have shown that permanental polynomials discriminate at least 5 graphs which have the same characteristic polynomials. However, since for trees, the coefficients of the permanental and characteristic polynomials differ in only signs, it is evident that permanental polynomials of isospectral trees will be the same.

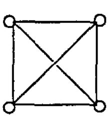

Fig. 4. A complete graph on 4 vertices (K_4)

Table 3. Imminant polynomials of K_n ($n = 4-7$)

Γ	x^4	x^2	x	1
		K_4		
[4]	1	6	−8	9
[31]	3	6	0	−9
[2^2]	2	0	8	6
[21^2]	3	−6	0	3
[1^4]	1	−6	−8	−3

Γ	x^5	x^3	x^2	x	1
			K_5		
[5]	1	10	−20	45	−44
[41]	4	20	−20	0	44
[32]	5	10	20	−15	−20
[31^2]	6	0	0	−30	−24
[2^21]	5	−10	20	45	20
[21^3]	4	−20	−20	0	4
[1^5]	1	−10	−20	−15	4

Γ	x^6	x^4	x^3	x^2	x	1
				K_6		
[6]	1	15	−40	135	−264	265
[51]	5	45	−80	135	0	−265
[42]	9	45	0	−45	144	135
[41^2]	10	30	−40	−90	120	130
[3^2]	5	15	40	45	−120	−55
[321]	16	0	80	0	−144	−80
[2^3]	5	−15	40	135	120	35
[31^3]	10	−30	−40	−90	−120	−50
[2^21^2]	9	−45	0	135	144	45
[21^4]	5	−45	−80	−45	0	5
[1^6]	1	−15	−40	−45	−24	−5

Γ	x^7	x^5	x^4	x^3	x^2	x	1
				K_7			
[7]	1	21	−70	315	−924	1855	−1854
[61]	6	84	−210	630	−924	0	1854
[52]	14	126	−140	210	−504	−910	−924
[51^2]	15	105	−210	105	420	−945	−930
[43]	14	84	70	−210	84	560	294
[421]	35	105	70	−305	420	1295	630
[3^21]	21	21	210	−105	−924	−945	−294
[41^3]	20	0	−140	−420	0	560	300
[32^2]	21	−21	−210	−305	84	−315	−126
[321^2]	35	−105	70	105	−420	−595	−210
[2^31]	14	−84	70	630	924	560	126
[31^4]	15	−105	−210	−315	−420	−315	−90
[2^21^3]	14	−126	−140	210	504	350	84
[21^5]	6	−84	−210	−210	−84	0	6
[1^7]	1	−21	−70	−105	−84	−35	−6

It would be interesting to explore if imminant polynomials will differentiate two isospectral non-tree graphs. At least in the chemistry literature, non-tree graphs are more common except in representing saturated alkanes for which, of course, trees are important. These are open questions which will be the topic of future investigations. Efficient and general algorithms are also desirable for the computation of imminant polynomials.

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